do not also show sensitive dependence on initial conditions. And indeed the pictures below illustrate that even in such cases changes in digit sequences are progressively amplified—just like in the shift map case (d).

But the crucial point that I will discuss more in Chapter 7 is that the presence of sensitive dependence on initial conditions in systems like (a) and (b) in no way implies that it is what is responsible for the randomness and complexity we see in these systems. And indeed, what looking at the shift map in terms of digit sequences shows us is that this phenomenon on its own can make no contribution at all to what we can reasonably consider the ultimate production of randomness.

Continuous Cellular Automata

Despite all their differences, the various kinds of programs discussed in the previous chapter have one thing in common: they are all based on elements that can take on only a discrete set of possible forms, typically just colors black and white. And in this chapter, we have introduced a similar kind of discreteness into our study of systems based on numbers.
by considering digit sequences in which each digit can again have only a
discrete set of possible values, typically just 0 and 1.

So now a question that arises is whether all the complexity we
have seen in the past three chapters somehow depends on the
discreteness of the elements in the systems we have looked at.

And to address this question, what I will do in this section is to
consider a generalization of cellular automata in which each cell is not
just black or white, but instead can have any of a continuous range of
possible levels of gray. One can update the gray level of each cell by
using rules that are in a sense a cross between the totalistic cellular
automaton rules that we discussed at the beginning of the last chapter
and the iterated maps that we just discussed in the previous section.

The idea is to look at the average gray level of a cell and its
immediate neighbors, and then to get the gray level for that cell at the
next step by applying a fixed mapping to the result. The picture below
shows a very simple case in which the new gray level of each cell is
exactly the average of the one for that cell and its immediate neighbors.
Starting from a single black cell, what happens in this case is that the
gray essentially just diffuses away, leaving in the end a uniform pattern.

A continuous cellular automaton in
which each cell can have any level of
gray between white (0) and black (1).
The rule shown here takes the new
gray level of each cell to be the average
of its own gray level and those of its
immediate neighbors.

The picture on the facing page shows what happens with a
slightly more complicated rule in which the average gray level is
multiplied by $3/2$, and then only the fractional part is kept if the result
of this is greater than 1.
And what we see is that despite the presence of continuous gray levels, the behavior that is produced exhibits the same kind of complexity that we have seen in many ordinary cellular automata and other systems with discrete underlying elements.
In fact, it turns out that in continuous cellular automata it takes only extremely simple rules to generate behavior of considerable complexity. So as an example the picture below shows a rule that determines the new gray level for a cell by just adding the constant $1/4$ to the average gray level for the cell and its immediate neighbors, and then taking the fractional part of the result.

The facing page and the one after show what happens when one chooses different values for the constant that is added. A remarkable diversity of behavior is seen. Sometimes the behavior is purely repetitive, but often it has features that seem effectively random. And in fact, as the picture in the middle of page 160 shows, it is even possible to find cases that exhibit localized structures very much like those occasionally seen in ordinary cellular automata.
More steps in the evolution of continuous cellular automata with the same kind of rules as on the previous page. In order to remove the uniform stripes, the picture in the middle shows the difference between the gray level of each cell and its immediate neighbor. Note the presence of discrete localized structures even though the underlying rules for the system involve continuous gray levels.