EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 4.7

Iterated Maps and the Chaos Phenomenon

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The basic idea of an iterated map is to take a number between 0 and 1, and then in a sequence of steps to update this number according to a fixed rule or "map". Many of the maps I will consider can be expressed in terms of standard mathematical functions, but in general all that is needed is that the map take any possible number between 0 and 1 and yield some definite number that is also between 0 and 1.

The pictures on the next two pages show examples of behavior obtained with four different possible choices of maps.

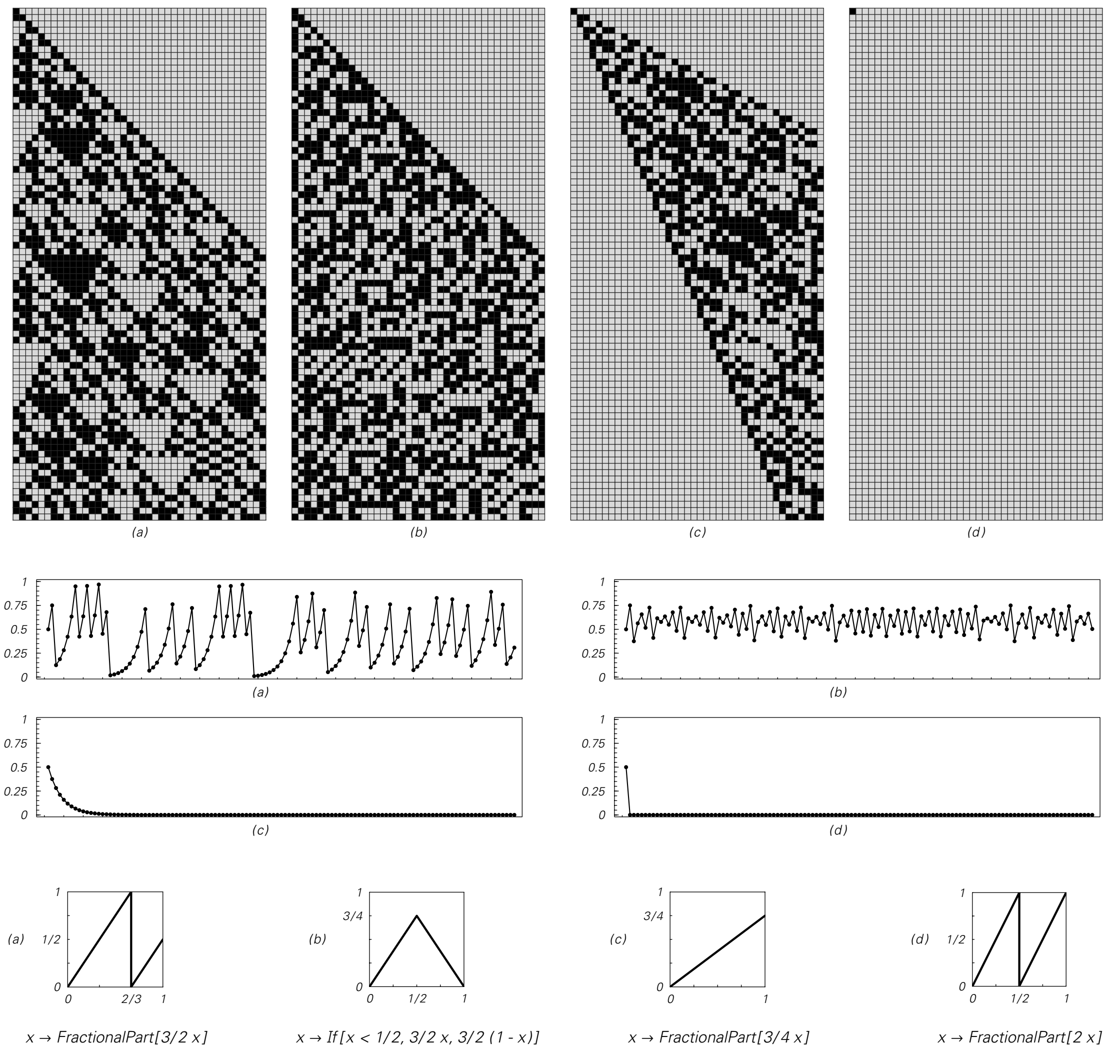
Cases (a) and (b) on the first page show much the same kind of complexity that we have seen in many other systems in this chapter—in both digit sequences and sizes of numbers. Case (c) shows complexity in digit sequences, but the sizes of the numbers it generates rapidly tend to 0. Case (d), however, seems essentially trivial—and shows no complexity in either digit sequences or sizes of numbers.

On the first of the next two pages all the examples start with the number 1/2—which has a simple digit sequence. But the examples on the second of the next two pages instead start with the number $\pi/4$—which has a seemingly random digit sequence.

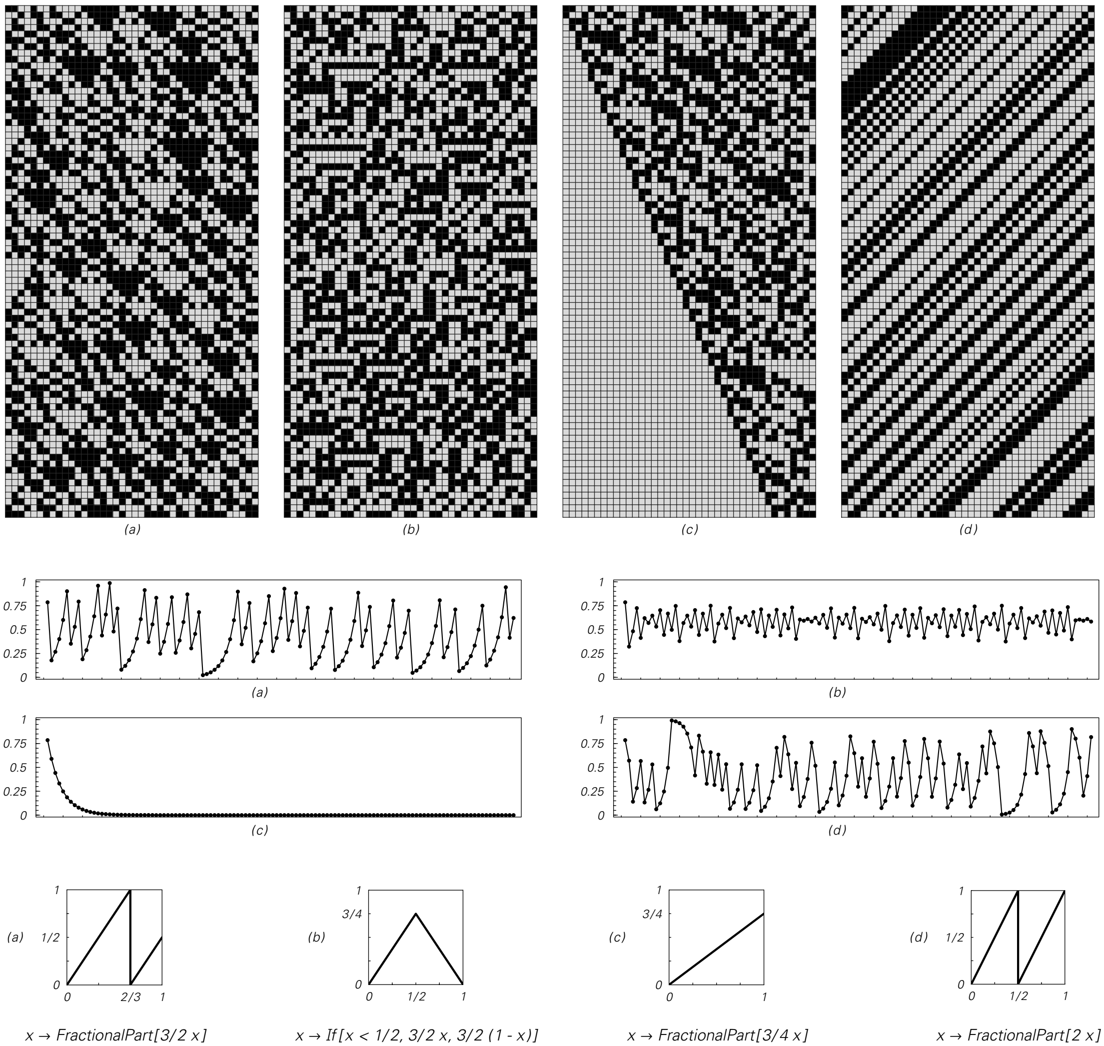
Cases (a), (b) and (c) look very similar on both pages, particularly in terms of sizes of numbers. But case (d) looks quite different. For on the first page it just yields 0's. But on the second page, it yields numbers whose sizes continually vary in a seemingly random way.

If one looks at digit sequences, it is rather clear why this happens. For as the picture illustrates, the so-called shift map used in case (d) simply serves to shift all digits one position to the left at each step. And this means that over the course of the evolution of the system, digits further to the right in the original number will progressively end up all the way to the left—so that insofar as these digits show randomness, this will lead to randomness in the sizes of the numbers generated.

It is important to realize, however, that in no real sense is any randomness actually being generated by the evolution of this system. Instead, it is just that randomness that was inserted in the digit sequence of the original number shows up in the results one gets.



Examples of iterated maps starting from simple initial conditions. At each step there is a number x between 0 and 1 that is updated by applying a fixed mapping. The four mappings considered here are given above both as formulas and in terms of plots. The pictures at the top of the page show the base 2 digit sequences of successive numbers obtained by iterating this mapping, while the pictures in the middle of the page plot the sizes of these numbers. In all cases, the initial conditions consist of the number 1/2—which has a very simple digit sequence. Yet despite this simplicity, cases (a) and (b) show considerable complexity in both the digit sequences and the sizes of the numbers produced (compare page 122). In case (c), the digit sequences are complicated but the sizes of the numbers tend rapidly to zero. And finally, in case (d), neither the digit sequences nor the sizes of numbers are anything but trivial. Note that in the pictures above each horizontal row of digits corresponds to a number, and that digits further to the left contribute progressively more to the size of this number.



The same iterated maps as on the facing page, but now started from the initial condition $\pi/4$—a number with a seemingly random digit sequence. After fairly few steps, cases (a) and (b) yield behavior that is almost indistinguishable from what was seen with simple initial conditions on the facing page. And in case (c), the same exponential decay in the sizes of numbers occurs as before. But in case (d), the behavior is much more complicated. Indeed, if one just looked at the sizes of numbers produced, then one sees the same kind of complexity as in cases (a) and (b). But looking at digit sequences one realizes that this complexity is actually just a direct transcription of complexity introduced by giving an initial condition with a seemingly random digit sequence. Case (d) is the so-called shift map—a classic example of a system that exhibits the sensitive dependence on initial conditions often known as chaos.

This is very different from what happens in cases (a) and (b). For in these cases complex and seemingly random results are obtained even on the first of the previous two pages—when the original number has a very simple digit sequence. And the point is that these maps actually do intrinsically generate complexity and randomness; they do not just transcribe it when it is inserted in their initial conditions.

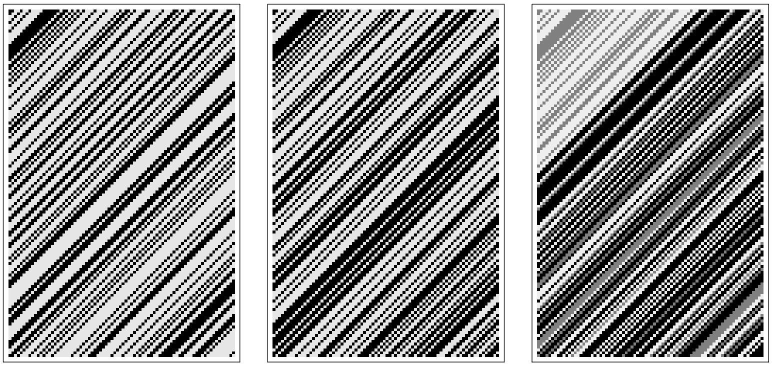
In the context of the approach I have developed in this book this distinction is easy to understand. But with the traditional mathematical approach, things can get quite confused. The main issue—already mentioned at the beginning of this chapter—is that in this approach the only attribute of numbers that is usually considered significant is their size. And this means that any issue based on discussing explicit digit sequences for numbers—and whether for example they are simple or complicated—tends to seem at best bizarre.

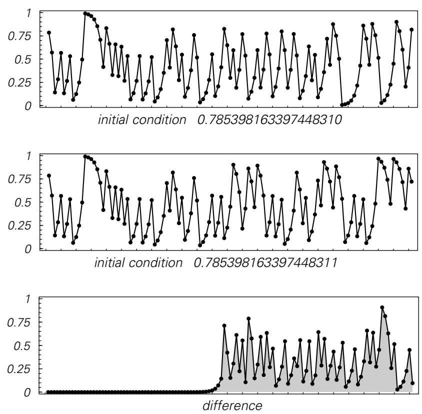
Indeed, thinking about numbers purely in terms of size, one might imagine that as soon as any two numbers are sufficiently close in size they would inevitably lead to results that are somehow also close. And in fact this is for example the basis for much of the formalism of calculus in traditional mathematics.

But the essence of the so-called chaos phenomenon is that there are some systems where arbitrarily small changes in the size of a number can end up having large effects on the results that are produced. And the shift map shown as case (d) on the previous two pages turns out to be a classic example of this.

The pictures at the top of the facing page show what happens if one uses as the initial conditions for this system two numbers whose sizes differ by just one part in a billion billion. And looking at the plots of sizes of numbers produced, one sees that for quite a while these two different initial conditions lead to results that are indistinguishably close. But at some point they diverge and soon become quite different.

And at least if one looks only at the sizes of numbers, this seems rather mysterious. But as soon as one looks at digit sequences, it immediately becomes much clearer. For as the pictures at the top of the facing page show, the fact that the numbers which are used as initial conditions differ only by a very small amount in size just means that their first several digits are the same. And for a while these digits are





The effect of making a small change in the initial conditions for the shift map—shown as case (d) on pages 150 and 151. The first picture shows results for the same initial condition as on page 151. The second picture shows what happens if one changes the size of the number in this initial condition by just one part in a billion billion. The plots to the left indicate that for a while the sizes of numbers obtained by the evolution of the system in these two cases are indistinguishable. But suddenly the results diverge and become completely different. Looking at the digit sequences above shows why this happens. The point is that a small change in the size of the number in the initial conditions corresponds to a change in digits far to the right. But the evolution of the system progressively shifts digits to the left, so that the digits which differ eventually become important. The much-investigated chaos phenomenon consists essentially of this effect.

what is important. But since the evolution of the system continually shifts digits to the left, it is inevitable that the differences that exist in later digits will eventually become important.

The fact that small changes in initial conditions can lead to large changes in results is a somewhat interesting phenomenon. But as I will discuss at length in Chapter 7 one must realize that on its own this cannot explain why randomness—or complexity—should occur in any particular case. And indeed, for the shift map what we have seen is that randomness will occur only when the initial conditions that are given happen to be a number whose digit sequence is random.

But in the past what has often been confusing is that traditional mathematics implicitly tends to assume that initial conditions of this kind are in some sense inevitable. For if one thinks about numbers purely in terms of size, one should make no distinction between numbers that are sufficiently close in size. And this implies that in choosing initial conditions for a system like the shift map, one should therefore make no distinction between the exact number 1/2 and numbers that are sufficiently close in size to 1/2.

But it turns out that if one picks a number at random subject only to the constraint that its size be in a certain range, then it is overwhelmingly likely that the number one gets will have a digit sequence that is essentially random. And if one then uses this number as the initial condition for a shift map, the results will also be correspondingly random—just like those on the previous page.

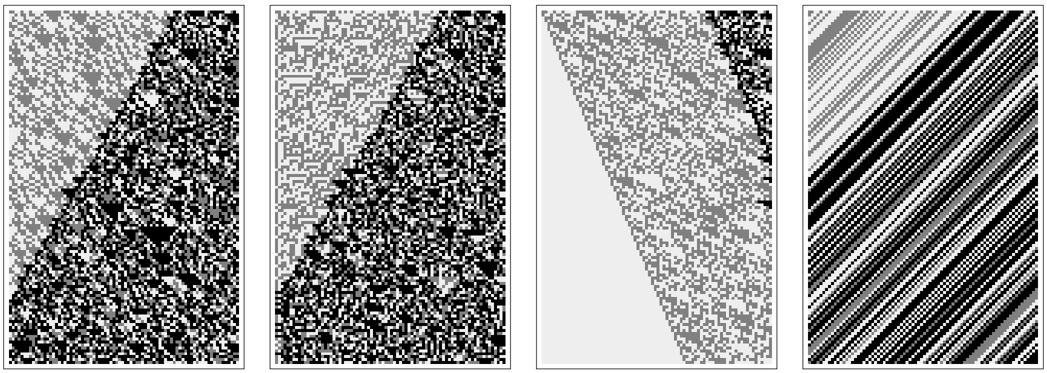
In the past this fact has sometimes been taken to indicate that the shift map somehow fundamentally produces randomness. But as I have discussed above, the only randomness that can actually come out of such a system is randomness that was explicitly put in through the details of its initial conditions. And this means that any claim that the system produces randomness must really be a claim about the details of what initial conditions are typically given for it.

I suppose in principle it could be that nature would effectively follow the same idealization as in traditional mathematics, and would end up picking numbers purely according to their size. And if this were so, then it would mean that the initial conditions for systems like the shift map would naturally have digit sequences that are almost always random.

But this line of reasoning can ultimately never be too useful. For what it says is that the randomness we see somehow comes from randomness that is already present—but it does not explain where that randomness comes from. And indeed—as I will discuss in Chapter 7—if one looks only at systems like the shift map then it is not clear any new randomness can ever actually be generated.

But a crucial discovery in this book is that systems like (a) and (b) on pages 150 and 151 can show behavior that seems in many respects random even when their initial conditions show no sign of randomness and are in fact extremely simple.

Yet the fact that systems like (a) and (b) can intrinsically generate randomness even from simple initial conditions does not mean that they do not also show sensitive dependence on initial conditions. And indeed the pictures below illustrate that even in such cases changes in digit sequences are progressively amplified—just like in the shift map case (d).



(a) (b) (c) (d)

Differences in digit sequences produced by a small change in initial conditions for the four iterated maps discussed in this section. Cases (a), (b) and (d) exhibit sensitive dependence on initial conditions, in the sense that a change in insignificant digits far to the right eventually grows to affect all digits. Case (c) does not show such sensitivity to initial conditions, but instead always evolves to 0, independent of its initial conditions.

But the crucial point that I will discuss more in Chapter 7 is that the presence of sensitive dependence on initial conditions in systems like (a) and (b) in no way implies that it is what is responsible for the randomness and complexity we see in these systems. And indeed, what looking at the shift map in terms of digit sequences shows us is that this phenomenon on its own can make no contribution at all to what we can reasonably consider the ultimate production of randomness.

Continuous Cellular Automata

Despite all their differences, the various kinds of programs discussed in the previous chapter have one thing in common: they are all based on elements that can take on only a discrete set of possible forms, typically just colors black and white. And in this chapter, we have introduced a similar kind of discreteness into our study of systems based on numbers