EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 4.5

Mathematical Constants

Mathematical Constants

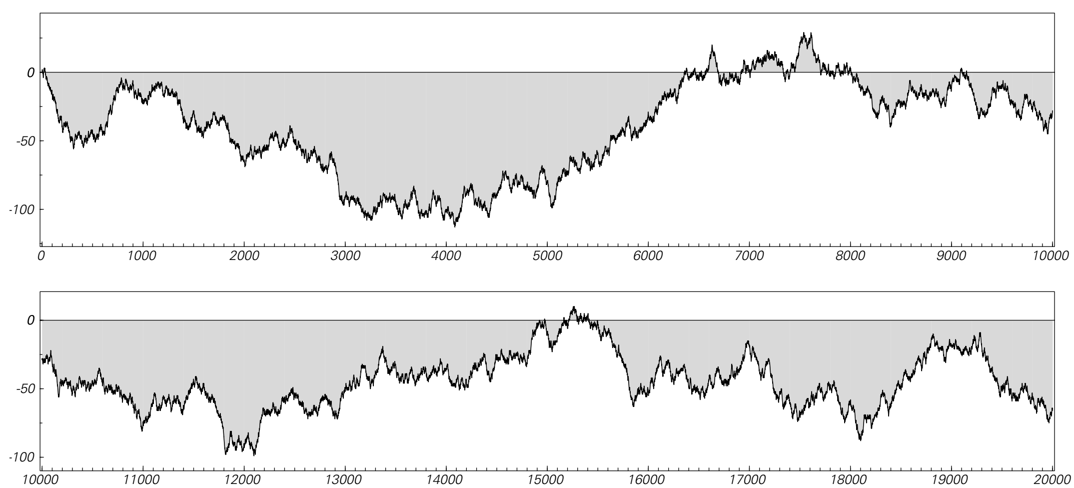
The last few sections have shown that one can set up all sorts of systems based on numbers in which great complexity can occur. But it turns out that the possibility of such complexity is already suggested by some well-known facts in elementary mathematics.

The facts in question concern the sequences of digits in numbers like π (pi). To a very rough approximation, π is 3.14. A more accurate approximation is 3.14159265358979323846264338327950288.

But how does this sequence of digits continue?

One might suppose that at some level it must be quite simple and regular. For the value of π is specified by the simple definition of being the ratio of the circumference of any circle to its diameter.

But it turns out that even though this definition is simple, the digit sequence of π is not simple at all. The facing page shows the first 4000 digits in the sequence, both in the usual case of base 10, and in base 2. And the picture below shows a pictorial representation of the first 20,000 digits in the sequence.



A pictorial representation of the first 20,000 digits of π in base 2. The curve drawn goes up every time a digit is 1, and down every time it is 0. Great complexity is evident. If the curve were continued further, it would spend more time above the axis, and no aspect of what is seen provides any evidence that the digit sequence is anything but perfectly random.

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3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384
  46095505822317253594081284811174502841027019385211055596446229489549303819644288109756659334461284756482337867831652712019091
  45648566923460348610454326648213393607260249141273724587006606315588174881520920962829254091715364367892590360011330530548820
  46652138414695194151160943305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912
  98336733624406566430860213949463952247371907021798609437027705392171762931767523846748184676694051320005681271452635608277857
  71342757789609173637178721468440901224953430146549585371050792279689258923542019956112129021960864034418159813629774771309960
  51870721134999999837297804995105973173281609631859502445945534690830264252230825334468503526193118817101000313783875288658753
  32083814206171776691473035982534904287554687311595628638823537875937519577818577805321712268066130019278766111959092164201989
  38095257201065485863278865936153381827968230301952035301852968995773622599413891249721775283479131515574857242454150695950829
  53311686172785588907509838175463746493931925506040092770167113900984882401285836160356370766010471018194295559619894676783744
  94482553797747268471040475346462080466842590694912933136770289891521047521620569660240580381501935112533824300355876402474964
  73263914199272604269922796782354781636009341721641219924586315030286182974555706749838505494588586926995690927210797509302955
  32116534498720275596023648066549911988183479775356636980742654252786255181841757467289097777279380008164706001614524919217321
  72147723501414419735685481613611573525521334757418494684385233239073941433345477624168625189835694855620992192221842725502542
  56887671790494601653466804988627232791786085784383827967976681454100953883786360950680064225125205117392984896084128488626945
  60424196528502221066118630674427862203919494504712371378696095636437191728746776465757396241389086583264599581339047802759009
  94657640789512694683983525957098258226205224894077267194782684826014769909026401363944374553050682034962524517493996514314298
  09190659250937221696461515709858387410597885959772975498930161753928468138268683868942774155991855925245953959431049972524680
  84598727364469584865383673622262609912460805124388439045124413654976278079771569143599770012961608944169486855584840635342207
  22258284886481584560285060168427394522674676788952521385225499546667278239864565961163548862305774564980355936345681743241125
  15076069479451096596094025228879710893145669136867228748940560101503308617928680920874760917824938589009714909675985261365549
  78189312978482168299894872265880485756401427047755513237964145152374623436454285844479526586782105114135473573952311342716610
  21359695362314429524849371871101457654035902799344037420073105785390621983874478084784896833214457138687519435064302184531910
  48481005370614680674919278191197939952061419663428754440643745123718192179998391015919561814675142691239748940907186494231961
  56794520809514655022523160388193014209376213785595663893778708303906979207734672218256259966150142150306803844773454920260541
  46659252014974428507325186660021324340881907104863317346496514539057962685610055081066587969981635747363840525714591028970641
  40110971206280439039759515677157700420337869936007230558763176359421873125147120532928191826186125867321579198414848829164470
  60957527069572209175671167229109816909152801735067127485832228718352093539657251210835791513698820914442100675103346711031412
  67111369908658516398315019701651511685171437657618351556508849099898599823873455283316355076479185358932261854896321329330898
  57064204675259070915481416549859461637180270981994309924488957571282890592323326097299712084433573265489382391193259746366730
  58360414281388303203824903758985243744170291327656180937734440307074692112019130203303801976211011004492932151608424448596376
  69838952286847831235526582131449576857262433441893039686426243410773226978028073189154411010446823252716201052652272111660397
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11.0010010000111111011010101000100010000101101000110000100011010011000100110001100110001010001011100000001101110000011100110100 0
  100101001000000010010011100000100010001010011001111100110001110100000000010000010111011111010100110001110110001001110011011001 0
  001001010001010010100000100001111001100011100011010000000010011011101111011110010101000110011011001111001101001110100100001 10
  001101100110000000101011000010100110110111110010010111110001010000110111010011111110000100110101011011010110110101010001110000
  100100010111100100100001011011010101110110011000100101111001111110110001101111010001001100010000101110100110100110001101111 11
  011010110101100001011111111110101110010110110111101000000011010110111111011011110111000111000011010111111101101011010100010 01
  100111111010010110101110100111110010010000001000101111100010010110001111111100110010010010010100001100110010100011110110011 100
  100010110110011110111000010000000000111110010111000101000010110001110111111100000101100110001101101001001000001101100001110 001
  010101110100111001101001101001000101100011111110101000111111010010010011001111010111111000001101100101010111010010001111011 10
  010100011101011011001011000011100011000101111001101010110001000001000010101010010101110111001111011010101001010010000011101 11
  000010010110100101100110110101100111000011000011010101001110010010101011110010011000000001001111000101110100011011000000100 01
  100101000011000001000010111110000110010100100000010111100100011000101110001101101100111000111011111000111001111001110111001 011
  000001100000000111010000110000000111001101100100111100000111010001011101100000000111101000101000111110110101110001010101110 1111
  100000110111110100110001010010110010011101111000101011110010111111011010010101010110000000101110001100000111001100101010100 1001
  011111001110101010010101011010101110010100010101110100100010011000011000100110001111101000000010100010000000010101011100101 0001
  110010110101000101010101010110001000010110110101101001100110001011100001101000000100010100000111101000110011101010000101010 100
  100001101010111101111110001110010111010011001001110110011111011100001010000001000101100011011011111011110000101010001010111 0101
  001110001010101110101110100000110000011000111110110110011100101110000111110000101101001101110000111100100110001111010101111 11
  010110101110100011001101101100001001001100111101011100011110100011001001010011100000001001010001001010110000110011101110011 101
  110001111010010001001100001101011010010111011100110101111110001001011111111101000000110110110011000101000001000011001001101 10
  000111011000000001001110011001111101100100001101010011001000101001000011111001010110001100000001011101111011001000000000011 00101
  110111110000100010111010101110111101001100001010111010110110001110111000010011000100011000000101110101101100101000110111000 10
  000001000111000100100111110100000001110100111001011010101100110001010000111101101101011011111111001110000011111101000100001 0001
  110010010111000001011010001001000000101010010010000100001000000000001000110100111001000111100000100101010011110000111111001 1011
  010111100010000111000110011010000100001011110110111010010110110010011010011001110000110010011100011000011010101111010011100 01
  000111100000110101001010001101000001101001011011000010101000010111101101000100101100000111110100111001010001010101101010001 00
  110011101000110110111011110111100001011011011000000100110111101000111011111001001011101000111011111100000101000001111110111 1101
  100101010100110001010000111110001011001010001110100111001101011110000000010111011001100110110010100101100100111110100000100 100
  001100001110100010001000110011101110100001100001100101000101011011111100111111011010001111110110000100101001011100001100111 0111
  000101101011110101111101110000001101111011101011101100010000101110000001001000000111001101000000000110100100010010011111010 101
  101100000010110101010100110010011101101001110101010011000100011011000111111011101110000011000011011111111101101111101110010 010
  000101001101100000001000111101001101111101000011010111001001001101000000000101000010010010010001101101100001111111010101101 0011
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The first 4000 digits of π in bases 10 and 2. Despite the simple definition of π as the ratio of the circumference to the diameter of a circle, its digit sequence is sufficiently complicated as to seem for practical purposes random.

In no case are there any obvious regularities. Indeed, in all the more than two hundred billion digits of π that have so far been computed, no significant regularity of any kind has ever been found. Despite the simplicity of its definition, the digit sequence of π seems for practical purposes completely random.

But what about other numbers? Is π a special case, or are there other familiar mathematical constants that have complicated digit sequences? There are some numbers whose digit sequences effectively have limited length. Thus, for example, the digit sequence of 3/8 in base 10 is 0.375. (Strictly, the digit sequence is 0.3750000000…, but the 0's do not affect the value of the number, so are normally suppressed.)

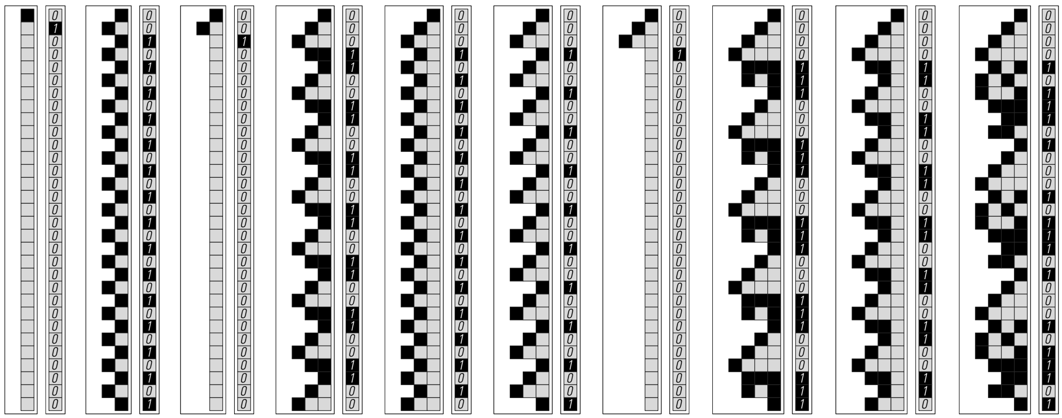
It is however easy to find numbers whose digit sequences do not terminate. Thus, for example, the exact value of 1/3 in base 10 is 0.3333333333333…, where the 3's repeat forever. And similarly, 1/7 is 0.142857142857142857142857142857…, where now the block of digits 142857 repeats forever. The table below gives the digit sequences for several rational numbers obtained by dividing pairs of whole numbers. In all cases what we see is that the digit sequences of such numbers have a simple repetitive form. And in fact, it turns out that absolutely all rational numbers have digit sequences that eventually repeat.

1/3 =	0.333333333333333333333333333333333333333333333333333333333333333333333333333…
1/7 =	0.142857142857142857142857142857142857142857142857142857142857142857142857142…
1/9 =	0.111111111111111111111111111111111111111111111111111111111111111111111111111…
1/11 =	0.090909090909090909090909090909090909090909090909090909090909090909090909090…
1/81 =	0.012345679012345679012345679012345679012345679012345679012345679012345679…

1/3 =	0.010101010101010101010101010101010101010101010101010101010101010101010101010…
1/7 =	0.001001001001001001001001001001001001001001001001001001001001001001001001001…
1/9 =	0.000111000111000111000111000111000111000111000111000111000111000111000111000…
1/11 =	0.000101110100010111010001011101000101110100010111010001011101000101110100010111010…
1/81 =	0.000000110010100100010110000111111001101011011101001111000000110010100100010110000…

Digit sequences for various rational numbers, given in base 10 (above) and base 2 (below). For a number of the form p/q, the digit sequence always repeats with a period of at most $q-1$ steps.

We can get some understanding of why this is so by looking at the details of how processes for performing division work. The pictures below show successive steps in a particular method for computing the base 2 digit sequence for the rational numbers p/q.



Successive steps in the computation of various rational numbers. In each case, the column on the right shows the sequence of base 2 digits in the number, while the box on the left shows the remainder at each of the steps in the computation.

The method is essentially standard long division, although it is somewhat simpler in base 2 than in the usual case of base 10. The idea is to have a number r which essentially keeps track of the remainder at each step in the division. One starts by setting r equal to p. Then at each step, one compares the values of $2r$ and q. If $2r$ is less than q, the digit generated at that step is 0, and r is replaced by $2r$. Otherwise, r is replaced by $2r - q$. With this procedure, the value of r is always less than q. And as a result, the digit sequence obtained always repeats at most every $q - 1$ steps.

It turns out, however, that rational numbers are very unusual in having such simple digit sequences. And indeed, if one looks for example at square roots the story is completely different.

Perfect squares such as $4 = 2\times2$ and $9 = 3\times3$ are specifically set up to have square roots that are just whole numbers. But as the table at the top of the next page shows, other square roots have much more complicated digit sequences. In fact, so far as one can tell, all whole numbers other than perfect squares have square roots whose digit sequences appear completely random.

$\sqrt{2}$ =	1.41421356237309504880168872420969807856967187537694807317667973799073247846210 7039...
$\sqrt{3}$ =	1.7320508075688772935274463415058723669428052538103806280558069794519330169088 00037...
$\sqrt{5}$ =	2.2360679774997896964091736687312762354406183596115257242708972454105209256378 04899...
$\sqrt{6}$ =	2.4494897427831780981972840747058913919659474806566701284326925672509603774573 15027...
$\sqrt{7}$ =	2.6457513110645905905016157536392604257102591830824501803683344592010688232302 83628...
$\sqrt{8}$ =	2.8284271247461900976033774484193961571393437507538961463533594759814649569242 14078...
$\sqrt{10}$ =	3.1622776601683793319988935444327185337195551393252168268575048527925944386392 38221...
$\sqrt{11}$ =	3.3166247903553998491149327366706866839270885455893535970586821461164846426090 43847...

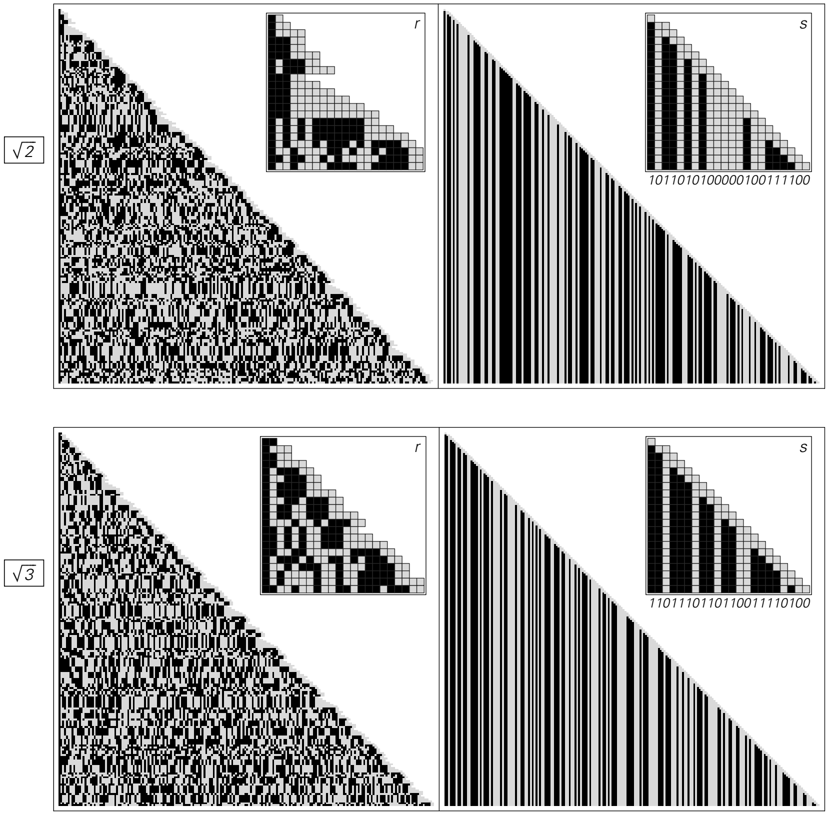
$\sqrt{2}$ =	1.0110101000001001111001100110011111110011101111001100100100001000101100101111110110...
$\sqrt{3}$ =	1.1011101101100111101011101000010110000100110010101010011100111011001001010111 01000...
$\sqrt{5}$ =	10.001111000110111011110011011100101111111010010100111110000010101111100111001 11001...
$\sqrt{6}$ =	10.011100110001000111000010100000001001001000010010111001111101000000110010000 110010...
$\sqrt{7}$ =	10.101001010100111111110101001110100101111100011101001101101111000111001110101 00111...
$\sqrt{8}$ =	10.110101000001001111001100110011111110011101111001100100100001000101100101111 110110...
$\sqrt{10}$ =	11.001010011000101100000111010110110100101101101010010100100100000001001010001 010111...
$\sqrt{11}$ =	11.010100010000111001010010011111111010110111100110100000101101000111011110010 01001...

Digit sequences for various square roots, given at the top in base 10 and at the bottom in base 2. Despite their simple definition, all these sequences seem for practical purposes random.

But how is such randomness produced? The picture at the top of the facing page shows an example of a procedure for generating the base 2 digit sequence for the square root of a given number n.

The procedure is only slightly more complicated than the one for division discussed above. It involves two numbers r and s, which are initially set to be n and 0, respectively. At each step it compares the values of r and s, and if r is larger than s it replaces r and s by $4(r-s-1)$ and $2(s+2)$ respectively; otherwise it replaces them just by $4r$ and $2s$. And it then turns out that the base 2 digits of s correspond exactly to the base 2 digits of $\sqrt{n}$—with one new digit being generated at each step.

As the picture shows, the results of the procedure exhibit considerable complexity. And indeed, it seems that just like so many other examples that we have discussed in this book, the procedure for generating square roots is based on simple rules but nevertheless yields behavior of great complexity.



A procedure for generating the digit sequences of square roots. Two numbers, r and s, are involved. To find $\sqrt{n}$ one starts by setting $r = n$ and $s = 0$. Then at each step one applies the rule $\{r, s\} \to If[r > s, \{4(r - s - 1), 2(s + 2)\}, \{4r, 2s\}]$. The result is that the digits of s in base 2 turn out to correspond exactly to the digits of $\sqrt{n}$. Note that if n is not between 1 and 4, it must be multiplied or divided by an appropriate power of 4 before starting this procedure.

It turns out that square roots are certainly not alone in having apparently random digit sequences. As an example, the table on the next page gives the digit sequences for some cube roots and fourth roots, as well as for some logarithms and exponentials. And so far as one can tell, almost all these kinds of numbers also have apparently random digit sequences.

In fact, rational numbers turn out to be the only kinds of numbers that have repetitive digit sequences. And at least in square roots, cube roots, and so on, it is known that no nested digit sequences

$\sqrt[3]{2}$	=	1.2599210498948731647672106072782283505702514647015079800819751121552996765139594837293965624362550941543102560 …
$\sqrt[3]{3}$	=	1.4422495703074083823216383107801095883918692534993505775464161945416875968299973398547554797056452566868350808 …
$\sqrt[4]{2}$	=	1.1892071150027210667174999705604759152929720924638174130190022247194666682269171598707813445381376737160373947 …
$\sqrt[4]{3}$	=	1.3160740129524924608192189017969990551600685902058221767319226585958667951973021330507431502466019315200477423 …
Log[2]	=	0 .6931471805599453094172321214581765680755001343602552541206800094933936219696947156058633269964186875420014810 …
Log[3]	=	1.0986122886681096913952452369225257046474905578227494517346943336374942932186089668736157548137320887879700290 …
e	=	2.7182818284590452353602874713526624977572470936999595749669676277240766303535475945713821785251664274274663919 …
e^2	=	7.3890560989306502272304274605750078131803155705518473240871278225225737960790577633843124850791217947737531612 …

$\sqrt[3]{2}$	=	1.01000001010001010001011111001100011010111001010001010111000100010001111011101101010110111000101011011111000100 …
$\sqrt[3]{3}$	=	1.0111000100110111010001001001000100100011111011110110010111001101110111100111111100010110110001010110111000110 …
$\sqrt[4]{2}$	=	1.00110000011011111111000001010001100011011011100010101001011011110100011010101101001000110001100000101110010000 …
$\sqrt[4]{3}$	=	1.0101000011101010001110011111110010111111000101100110010111110110110111000011001100111111000101000001101001101 …
Log[2]	=	0 .10110001011100100001011111110111110100011100111101111001101010111100100111100011101100111001100000000011111100 …
Log[3]	=	1.0001100100111110101001111010101011010000001100001010100101110110101001000001100110001101010101010000010100111 …
e	=	10.1011011111100001010100010110001010001010111011010010101001101010101111110111000101011000100000000100111001111 …
e^2	=	111.01100011100110010010111000110101001101110110101101110011000011001110100011101110100010000001101011011010001 …

Digit sequences for cube roots, fourth roots, logarithms and exponentials, given at the top in base 10 and the bottom in base 2. Once again, these sequences seem for practical purposes random.

ever occur. It is straightforward to construct a nested digit sequence using for example the substitution systems on page 83, but the point is that such a digit sequence never corresponds to a number that can be obtained by the mathematical operation of taking roots.

So far in this chapter we have always used digit sequences as our way of representing numbers. But one might imagine that perhaps this representation is somehow perverse, and that if we were just to choose another one, then numbers generated by simple mathematical operations would no longer seem complex.

Any representation for a number can in a sense be thought of as specifying a procedure for constructing that number. Thus, for example, the pictures at the top of the facing page show how the base 10 and base 2 digit sequence representations of π can be used to construct the number π.

$$3.141592653\ldots = 3 + \frac{1}{10}\left(1 + \frac{1}{10}\left(4 + \frac{1}{10}\left(1 + \frac{1}{10}\left(5 + \frac{1}{10}\left(9 + \frac{1}{10}\left(2 + \frac{1}{10}\left(6 + \frac{1}{10}\left(5 + \frac{1}{10}\left(3 + \ldots\right)\right)\right)\right)\right)\right)\right)\right)\right)$$

$$11.001001000\ldots = 2 + 1 + \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{2}\left(1 + \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{2}\left(1 + \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \ldots\right)\right)\right)\right)\right)\right)\right)\right)\right)$$

Procedures for building up π from its base 10 and base 2 digit sequence representations.

By replacing the addition and multiplication that appear above by other operations one can then get other representations for numbers. A common example are so-called continued fraction representations, in which the operations of addition and division are used, as shown below.

$$3 + 1/(7 + 1/(15 + 1/(1 + 1/(292 + 1/(1 + 1/(1 + 1/(1 + 1/(2 + 1/(1 + 1/(3 + 1/(1 + 1/(14 + \ldots))))))))))))$$

$$\{3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, \ldots\}$$

The continued fraction representation of π. In this representation the value of π is built up by successive additions and divisions, rather than successive additions and multiplications.

The table on the next page gives the continued fraction representations for various numbers. In the case of rational numbers, the results are always of limited length. But for other numbers, they go on forever. Square roots turn out to have purely repetitive continued fraction representations. And the representations of $e \simeq 2.718$ and all its roots also show definite regularity. But for π, as well as for cube roots, fourth roots, and so on, the continued fraction representations one gets seem essentially random.

What about other representations of numbers? At some level, one can always use symbolic expressions like $\sqrt{2} + e^{\sqrt{3}}$ to represent numbers. And almost by definition, numbers that can be obtained by simple mathematical operations will correspond to simple such expressions. But the problem is that there is no telling how difficult it may be to compute the actual value of a number from the symbolic expression that is used to represent it.

And in thinking about representations of numbers, it seems appropriate to restrict oneself to cases where the effort required to find the value of a number from its representation is essentially the same for

$1/7 = \{0, 7\}$
$7/11 = \{0, 1, 1, 1, 3\}$
$\sqrt{2} = \{1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots\}$
$\sqrt{3} = \{1, 1, 2, 1, \ldots\}$
$\sqrt{5} = \{2, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \ldots\}$
$\sqrt{7} = \{2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, \ldots\}$
$(1+\sqrt{5})/2 = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots\}$
$\sqrt[3]{2} = \{1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1, 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, 121, 1, 2, 2, 4, 10, 3, 2, 2, \ldots\}$
$\sqrt[3]{3} = \{1, 2, 3, 1, 4, 1, 5, 1, 1, 6, 2, 5, 8, 3, 3, 4, 2, 6, 4, 4, 1, 3, 2, 3, 4, 1, 4, 9, 1, 8, 4, 3, 1, 3, 2, 6, 1, 6, 1, 3, 1, 1, 1, 1, 12, 3, 1, 3, 1, 1, 4, 1, 6, 1, 5, \ldots\}$
$\sqrt[4]{2} = \{1, 5, 3, 1, 1, 40, 5, 1, 1, 25, 2, 3, 1, 6, 2, 1, 1, 2, 1, 2, 1, 1, 1, 2, 2, 1, 7, 2, 7, 1, 1, 1, 2, 1, 1, 32, 4, 1, 6, 2, 1, 1, 1, 15, 1, 5, 1, 4, 1, 1, 1, 3, 1, 3, \ldots\}$
$\sqrt[4]{3} = \{1, 3, 6, 9, 1, 1, 2, 1, 2, 1, 2, 5, 1, 12, 5, 1, 4, 1, 13, 1, 6, 1, 22, 1, 8, 21, 3, 142, 1, 1, 2, 1, 2, 2, 7, 1, 2, 1, 1, 1, 5, 3, 1, 1, 2, 1, 1, 3, 1, 1, 1, 1, \ldots\}$
$Log[2] = \{0, 1, 2, 3, 1, 6, 3, 1, 1, 2, 1, 1, 1, 1, 3, 10, 1, 1, 1, 2, 1, 1, 1, 1, 3, 2, 3, 1, 13, 7, 4, 1, 1, 1, 7, 2, 4, 1, 1, 2, 5, 14, 1, 10, 1, 4, 2, 18, 3, 1, 4, 1, 6, \ldots\}$
$Log[3] = \{1, 10, 7, 9, 2, 2, 1, 3, 1, 32, 2, 17, 1, 15, 1, 1, 7, 3, 1, 35, 1, 1, 1, 2, 5, 3, 2, 1, 4, 2, 1, 3, 1, 5, 3, 13, 1, 1, 1, 6, 2, 3, 1, 152, 1, 2, 3, 1, 7, 9, 2, \ldots\}$
$e = \{2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, 1, 1, 22, 1, 1, 24, 1, 1, 26, 1, 1, 28, 1, 1, 30, 1, 1, 32, 1, \ldots\}$
$\sqrt{e} = \{1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, 1, 1, 17, 1, 1, 21, 1, 1, 25, 1, 1, 29, 1, 1, 33, 1, 1, 37, 1, 1, 41, 1, 1, 45, 1, 1, 49, 1, 1, 53, 1, 1, 57, 1, 1, 61, 1, 1, \ldots\}$
$\sqrt[3]{e} = \{1, 2, 1, 1, 8, 1, 1, 14, 1, 1, 20, 1, 1, 26, 1, 1, 32, 1, 1, 38, 1, 1, 44, 1, 1, 50, 1, 1, 56, 1, 1, 62, 1, 1, 68, 1, 1, 74, 1, 1, 80, 1, 1, 86, 1, 1, 92, 1, 1, \ldots\}$
$e^2 = \{7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, 1, 1, 12, 54, 14, 1, 1, 15, 66, 17, 1, 1, 18, 78, 20, 1, 1, 21, 90, 23, 1, 1, 24, 102, 26, 1, 1, 27, \ldots\}$
$e^3 = \{20, 11, 1, 2, 4, 3, 1, 5, 1, 2, 16, 1, 1, 16, 2, 13, 14, 4, 6, 2, 1, 1, 2, 2, 2, 3, 5, 1, 3, 1, 1, 68, 7, 5, 1, 4, 2, 1, 1, 1, 1, 1, 1, 7, 3, 1, 6, 1, 2, 5, 4, 7, \ldots\}$
$\pi = \{3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, \ldots\}$
$\pi^2 = \{9, 1, 6, 1, 2, 47, 1, 8, 1, 1, 2, 2, 1, 1, 8, 3, 1, 10, 5, 1, 3, 1, 2, 1, 1, 3, 15, 1, 1, 2, 2, 1, 3, 2, 7, 1, 9, 18, 30, 2, 145, 1, 1, 17, 9, 1, 1, 1, 1, 7, 12, 1, \ldots\}$
$Sinh[1] = \{1, 5, 1, 2, 2, 2, 1, 2, 7, 5, 1, 1, 1, 2, 2, 19, 1, 2, 1, 7, 1, 1, 9, 1, 3, 1, 1, 2, 1, 1, 1, 1, 1, 3, 1, 2, 4, 5, 3, 5, 1, 3, 1, 1, 1, 2, 7, 1, 9, 1, 1, 2, 1, 21, 1, \ldots\}$
$Tanh[1] = \{0, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, \ldots\}$

Continued fraction representations for several numbers. Square roots yield repetitive sequences in this representation, but cube roots and higher roots yield seemingly random sequences.

all numbers. If one does this, then the typical experience is that in any particular representation, some class of numbers will have simple forms. But other numbers, even though they may be the result of simple mathematical operations, tend to have seemingly random forms.

And from this it seems appropriate to conclude that numbers generated by simple mathematical operations are often in some intrinsic sense complex, independent of the particular representation that one uses to look at them.