EXCERPTED FROM

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# A NEW KIND OF SCIENCE

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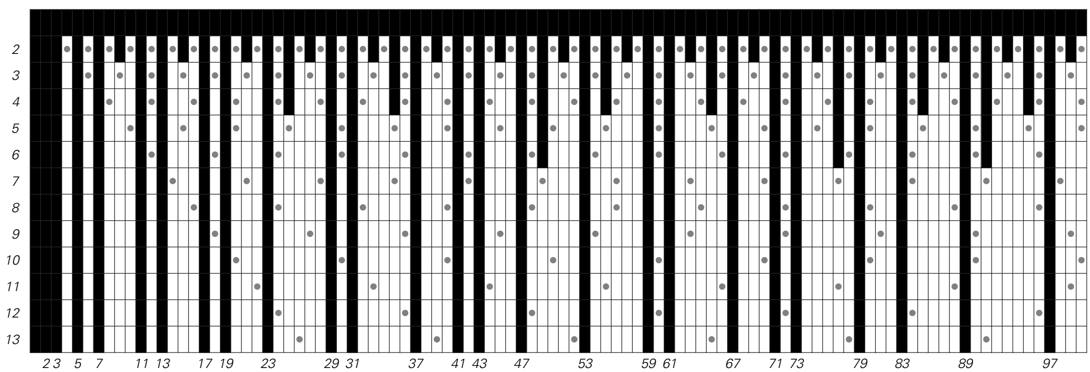
SECTION 4.4

## *The Sequence of Primes*

## The Sequence of Primes

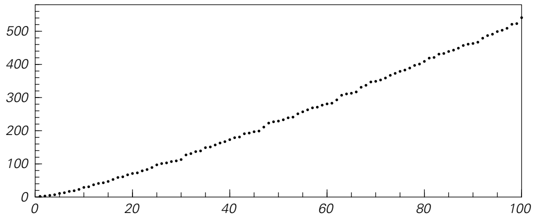
In the sequence of all possible numbers 1, 2, 3, 4, 5, 6, 7, 8, ... most are divisible by others—so that for example 6 is divisible by 2 and 3. But this is not true of every number. And so for example 5 and 7 are not divisible by any other numbers (except trivially by 1). And in fact it has been known for more than two thousand years that there are an infinite sequence of so-called prime numbers which are not divisible by other numbers, the first few being 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ...

The picture below shows a simple rule by which such primes can be obtained. The idea is to start out on the top line with all possible numbers. Then on the second line, one removes all numbers larger than 2 that are divisible by 2. On the third line one removes numbers divisible by 3, and so on. As one goes on, fewer and fewer numbers remain. But some numbers always remain, and these numbers are exactly the primes.

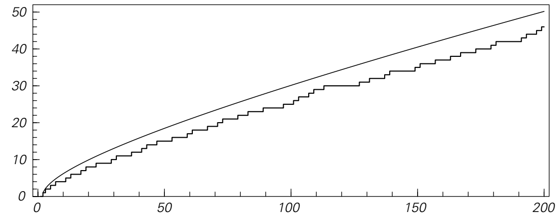



A filtering process that yields the prime numbers. One starts on the top line with all numbers between 1 and 100. Then on the second line, one removes numbers larger than 2 that are divisible by 2—as indicated by the gray dots. On the third line, one removes numbers larger than 3 that are divisible by 3. If one then continues forever, there are some numbers that always remain, and these are exactly the primes. The process shown is essentially the sieve of Eratosthenes, already known in 200 BC.

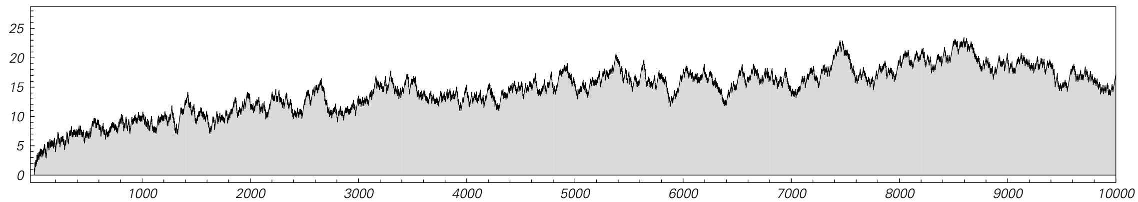
Given the simplicity of this rule, one might imagine that the sequence of primes it generates would also be correspondingly simple. But just as in so many other examples in this book, in fact it is not. And indeed the plots on the facing page show various features of this sequence which indicate that it is in many respects quite random.



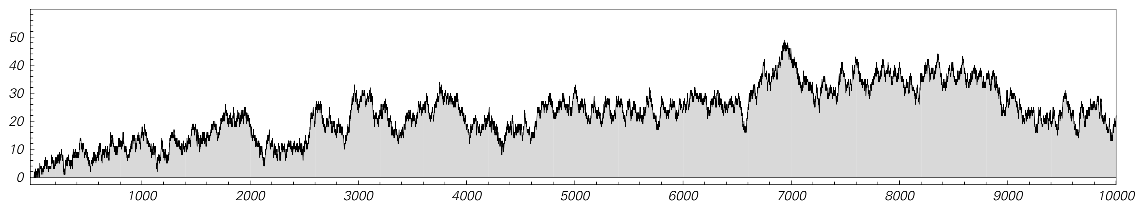

(a) The sequence of primes (Prime[n])



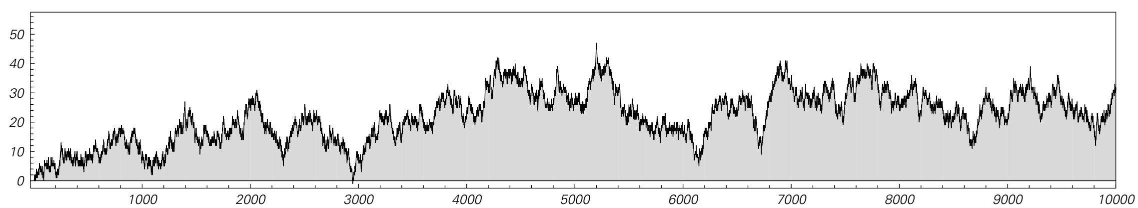

(b) The number of primes smaller than n (PrimePi[n]), together with the estimate LogIntegral[n]



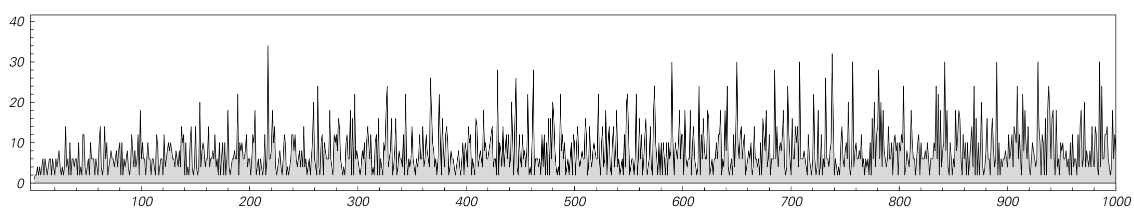

(c) The difference LogIntegral[n] – PrimePi[n]




(d) The excess of primes of the form 3 k – 1 over ones of the form 3 k + 1




(e) The excess of primes of the form 4 k – 1 over ones of the form 4 k + 1




(f) Gaps between successive primes

Features of the sequence of primes. Despite the simplicity of the rule on the facing page that generates the primes, the actual sequence of primes that is obtained seems in many respects remarkably random.

The examples of complexity that I have shown so far in this book are almost all completely new. But the first few hundred primes were no doubt known even in antiquity, and it must have been evident that there was at least some complexity in their distribution.

However, without the whole intellectual structure that I have developed in this book, the implications of this observation—and its potential connection, for example, to phenomena in nature—were not recognized. And even though there has been a vast amount of mathematical work done on the sequence of primes over the course of many centuries, almost without exception it has been concerned not with basic issues of complexity but instead with trying to find specific kinds of regularities.

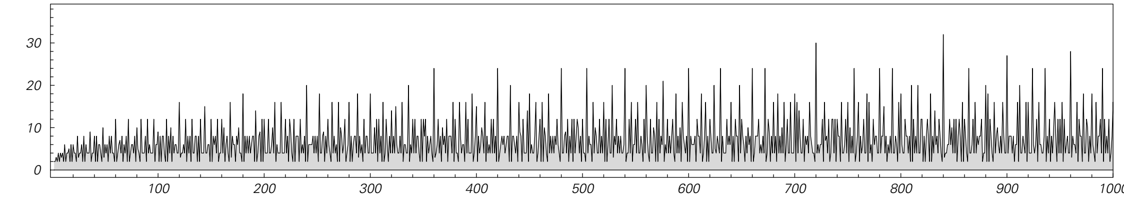
Yet as it turns out, few regularities have in fact been found, and often the results that have been established tend only to support the idea that the sequence has many features of randomness. And so, as one example, it might appear from the pictures on the previous page that (c), (d) and (e) always stay systematically above the axis. But in fact with considerable effort it has been proved that all of them are in a sense more random—and eventually cross the axis an infinite number of times, and indeed go any distance up or down.

So is the complexity that we have seen in the sequence of primes somehow unusual among sequences based on numbers? The pictures on the facing page show a few other examples of sequences generated according to simple rules based on properties of numbers.

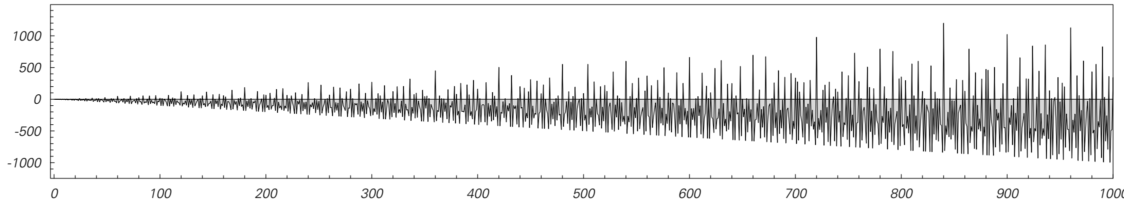
And in each case we again see a remarkable level of complexity.

Some of this complexity can be understood if we look at each number not in terms of its overall size, but rather in terms of its digit sequence or set of possible divisors. But in most cases—often despite centuries of work in number theory—considerable complexity remains.

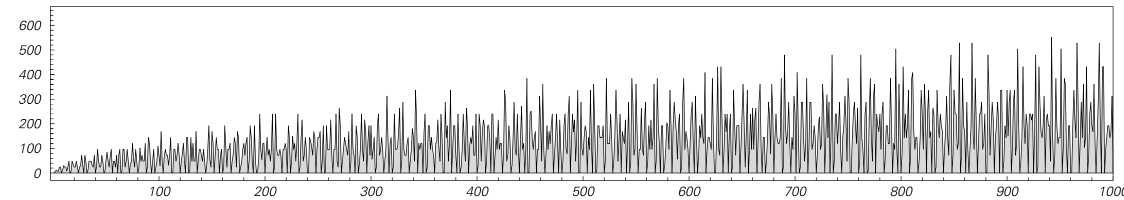
And indeed the only reasonable conclusion seems to be that just as in so many other systems in this book, such sequences of numbers exhibit complexity that somehow arises as a fundamental consequence of the rules by which the sequences are generated.



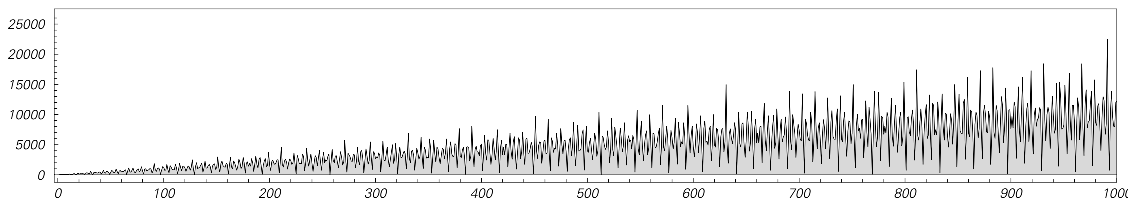

(a) The number of divisors of n (including n)



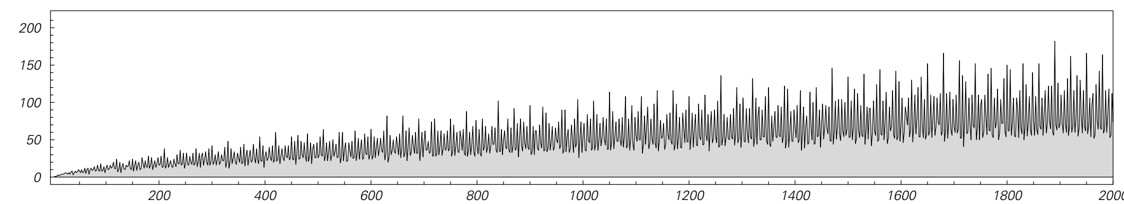

(b) The sum of the divisors of n (excluding n) minus n




(c) The number of ways of expressing n as a sum of three squares




(d) The number of ways of expressing n as a sum of four squares




(e) The number of ways of expressing an even number n as the sum of two primes

Sequences based on various simple properties of numbers. Extensive work in number theory has managed to establish only a few properties of these. It is for example known that (d) never reaches zero, while curve (c) reaches zero only for numbers of the form $4^r(8s+7)$. Sequence (b) is zero at so-called perfect numbers. Even perfect numbers always have a known form, but whether any odd perfect numbers exist is a question that has remained unresolved for more than two thousand years. The claim that sequence (e) never reaches zero is known as Goldbach's Conjecture. It was made in 1742 but no proof or counterexample has ever been found.